

Bound-States of the Spinless Salpeter Equation for the \mathcal{PT} -Symmetric Generalized Hulthén Potential by the Nikiforov-Uvarov Method

Sameer M. Ikhdair* and Ramazan Sever†

**Department of Physics, Near East University, Nicosia, North
Cyprus, Mersin 10, Turkey*

*†Department of Physics, Middle East Technical University, 06531
Ankara, Turkey.*

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Abstract

The one-dimensional spinless Salpeter equation has been solved for the \mathcal{PT} -symmetric generalized Hulthén potential. The Nikiforov-Uvarov (NU) method which is based on solving the second-order linear differential equations by reduction to a generalized equation of hypergeometric type is used to obtain exact energy eigenvalues and corresponding eigenfunctions. We have investigated the positive and negative exact bound states of the s-states for different types of complex generalized Hulthén potentials.

Keywords: Bethe-Salpeter equation, Energy Eigenvalues and Eigenfunctions; Generalized Hulthén potential; \mathcal{PT} -symmetry, NU Method.

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*sameer@neu.edu.tr

†sever@metu.edu.tr

I. INTRODUCTION

In the past few years there has been considerable work on non-Hermitian Hamiltonians. Among this kind of Hamiltonians, much attention has been focused on the investigation of properties of so-called \mathcal{PT} -symmetric Hamiltonians. Following the early studies of Bender *et al.* [1], the \mathcal{PT} -symmetry formulation has been successfully utilized by many authors [2-8]. The \mathcal{PT} -symmetric but non-Hermitian Hamiltonians have real spectra whether the Hamiltonians are Hermitian or not. Non-Hermitian Hamiltonians with real or complex spectra have also been analyzed by using different methods [3-6,9]. Non-Hermitian but \mathcal{PT} -symmetric models have applications in different fields, such as optics [10], nuclear physics [11], condensed matter [12], quantum field theory [13] and population biology [14].

Exact solution of Schrödinger equation for central potentials has generated much interest in recent years. So far, some of these potentials are the parabolic type potential [15], the Eckart potential [16,17], the Fermi-step potential [16,17], the Rosen-Morse potential [18], the Ginocchio barrier [19], the Scarf barriers [20], the Morse potential [21] and a potential which interpolates between Morse and Eckart barriers [22]. Many authors have studied on exponential type potentials [23,24,25,26] and quasi exactly solvable quadratic potentials [27,28,29]. In addition, Schrödinger, Dirac, Klein-Gordon, and Duffin-Kemmer-Petiau equations for a Coulomb type potential are solved by using different method [30,31,32,33,34]. The exact solutions for these models have been obtained analytically.

Further, using the quantization of the boundary condition of the states at the origin, Znojil [35] studied another generalized Hulthén and other exponential potentials in non-relativistic and relativistic regions. Domingues-Adame [36] and Chetouani *et al.* [37] also studied relativistic bound states of the standard Hulthén potential. On the other hand, Rao and Kagali [38] investigated the relativistic bound states of the exponential-type screened Coulomb potential by means of the one-dimensional Klein-Gordon equation. However, it is well-known that for the exponential-type screened Coulomb potential there is no explicit form of the energy expression of bound-states for Schrödinger [39], KG [38] and also Dirac

[16] equations. In a recent work [31], Şimşek and Eğřifes have presented the bound-state solutions of one-dimensional (1D) Klein-Gordon equation for \mathcal{PT} -symmetric potentials with real and complex forms of the generalized Hulthén potential. In a latter study [32], Eğřifes and Sever investigated the bound-state solutions of the 1D Dirac equation for real and complex forms of generalized Hulthén potential for \mathcal{PT} -symmetric potentials with complex generalized Hulthén potential. In recent works, we have solved the 1D Schrödinger equation with the \mathcal{PT} -symmetric modified Hulthén and Woods-Saxon (WS) potentials for $\ell \neq 0$ bound-state spectra and their corresponding wave functions [40,41] using the Nikiforov-Uvarov (NU) method [42]. In the latter case, we investigated the \mathcal{PT} -symmetric property and the reality of the spectrum for different real and complex versions of the modified WS potentials [41].

In the present study we investigate the bound-states solutions of the 1D spinless Salpeter (SS) equation for the real and complex forms of the generalized Hulthén potential and corresponding eigen functions using the same method. Thus, the objective of this work is to deal with the \mathcal{PT} -symmetric property and the existence of bound-states (i.e., the reality of energy spectrum) when we solve the spinless Salpeter (SS) equation for some real and complex potentials with standard generalized forms. In view of the \mathcal{PT} -symmetric formulation, we shall apply the NU method to solve the s -wave SS equation. In this regard, it is possible to present the theory of special functions by starting from a differential equation which is based on solving the SS equation by reducing it into a generalized equation of hypergeometric form. We also seek to present exact bound states for a family of exponential-type potentials, i.e., generalized Hulthén potential which is reducible to the standard Hulthén potential, Woods-Saxon potential and exponential-type screened Coulomb potential. This family of potentials have been applied with success to a number of different fields of physical systems.

The organization of the present work is as follows. After a brief introductory discussion of the NU method in Section II, we obtain the bound-state energy spectra for real and complex cases of generalized Hulthén potentials and their corresponding eigenfunctions in

Section III. Finally, we end with some results and conclusions in Section IV.

II. THE NIKIFOROV-UVAROV METHOD

In this section we outline the basic formulations of the method. The Schrödinger equation and other Schrödinger-type equations can be solved by using the Nikiforov-Uvarov (NU) method which is based on the solutions of general second-order linear differential equation with special orthogonal functions [42]. It is well known that for any given 1D radial potential, the Schrödinger equation can be reduced to a generalized equation of hypergeometric type with an appropriate transformation and it can be written in the following form

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \quad (1)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is of a first-degree polynomial. To find particular solution of Eq.(1) we apply the method of separation of variables using the transformation

$$\psi_n(s) = \phi_n(s)y_n(s), \quad (2)$$

which reduces equation (1) into a hypergeometric-type equation

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \quad (3)$$

whose polynomial solutions $y_n(s)$ of the hypergeometric type function are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \quad (n = 0, 1, 2, \dots) \quad (4)$$

where B_n is a normalizing constant and $\rho(s)$ is the weight function satisfying the condition [42]

$$\frac{d}{ds}w(s) = \frac{\tau(s)}{\sigma(s)}w(s), \quad (5)$$

where $w(s) = \sigma(s)\rho(s)$. On the other hand, the function $\phi(s)$ satisfies the condition

$$\frac{d}{ds}\phi(s) = \frac{\pi(s)}{\sigma(s)}\phi(s), \quad (6)$$

where the linear polynomial $\pi(s)$ is given by

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \quad (7)$$

from which the root k is the essential point in the calculation of $\pi(s)$ is determined. Further, the parameter λ required for this method is defined as

$$\lambda = k + \pi'(s). \quad (8)$$

Further, in order to find the value of k , the discriminant under the square root is being set equal to zero and the resulting second-order polynomial has to be solved for its roots $k_{+,-}$. Thus, a new eigenvalue equation for the SE becomes

$$\lambda_n + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s) = 0, \quad (n = 0, 1, 2, \dots) \quad (9)$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad (10)$$

and it must have a negative derivative.

III. RADIAL SPINLESS SALPETER EQUATION

The relativistic wave Salpeter equation [43] is constructed by considering the kinetic energies of the constituents and the interaction potential. In addition, the spinless Salpeter (SS) for the case of two particles with unequal masses m_1 and m_2 , interacting by a spherically symmetric potential $V(r)$ in the center-of-momentum system of the two particles is given by

$$\left[\sum_{i=1,2} \sqrt{-\Delta + m_i^2} + V(r) - M \right] \chi(\mathbf{r}) = 0, \quad \Delta = \nabla^2 \quad (11)$$

where the kinetic terms involving the operation $\sqrt{-\Delta + m_i^2}$ are nonlocal operators and $\chi(\mathbf{r}) = Y_{\ell,m}(\theta, \phi) R_{n,l}(\mathbf{r})$ denotes the Salpeter's wave function. For heavy interacting particles, the kinetic energy operators in Eq. (11) can be approximated, (cf. e.g., Jaczko and Durand [44], Ikhdair and Sever [45,46]), as

$$\sum_{i=1,2} \sqrt{-\Delta + m_i^2} = m_1 + m_2 - \frac{\Delta}{2\mu} - \frac{\Delta^2}{8\eta^3} - \dots, \quad (12)$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ stands for the reduced mass and $\eta = \mu \left(\frac{m_1 m_2}{m_1 m_2 - 3\mu^2} \right)^{1/3}$ is an introduced useful mass parameter. This SS-type equation retains its relativistic kinematics and is suitable for describing the spin-averaged spectrum of two bound particles of masses m_1 and m_2 . The Hamiltonian containing the relativistic corrections up to order (v^2/c^2) is called a generalized Breit-Fermi Hamiltonian (cf. e.g., Lucha *et al.* [47]). Hence, the SS equation can be further written in the form ($\hbar = c = 1$) [45,46]

$$\left\{ -\frac{\Delta}{2\mu} - \frac{\Delta^2}{8\eta^3} + V(r) \right\} R_{nl}(\mathbf{r}) = E_{nl} R_{nl}(\mathbf{r}), \quad (13)$$

where $E_{nl} = M_{nl} - m_1 - m_2$ refers to the Salpeter binding energy with M_{nl} is the semirelativistic-bound-state masses.¹ Moreover, it is worthwhile to point out here that, to obtain a Schrödinger-type equation, as could be seen from Eq.(13), the perturbation term can be treated by using the reduced Schrödinger equation [49]

$$p^4 = 4\mu^2 [E_{nl} - V(r)]^2, \quad (14)$$

with $p^4 = \Delta^2$, and consequently one would reduce Eq. (13) into a Schrödinger-type form [45,46]

$$\left\{ -\frac{\Delta}{2\mu} - \frac{\mu^2}{2\eta^3} [E_{nl}^2 + V^2(r) - 2E_{nl}V(r)] + V(r) \right\} R_{nl}(\mathbf{r}) = E_{nl} R_{nl}(\mathbf{r}). \quad (15)$$

¹This approximation is correct to $O(v^2/c^2)$. The Δ^2 term in (13) should be properly treated as a perturbation by using trial wavefunctions [48].

In addition, the three-dimensional ($3D$) space operator in the spherical polar coordinates takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{L^2}{r^2}, \quad (16)$$

with $L^2 = l(l+1)$. Hence, after employing the following transformation

$$R_{nl}(r) = \frac{\psi_{nl}(r)}{r}, \quad (17)$$

we obtain [45]

$$\Delta = \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}, \quad \Delta^2 = \frac{\partial^4}{\partial r^4} - \frac{2l(l+1)}{r^2} \frac{\partial^2}{\partial r^2} + \frac{4l(l+1)}{r^3} \frac{\partial}{\partial r} + \frac{l(l+1)[l^2+l-6]}{r^4}. \quad (18)$$

Thus, using Eqs (17) and (18) and after a lengthy algebra but straightforward, we can finally write Eq.(15) in the following $3D$ space

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + W_{nl}(r) - \frac{W_{nl}(r)^2}{2\widetilde{m}} \right] \psi_{nl}(r) = 0, \quad (19)$$

where

$$W_{nl}(r) = V(r) - E_{nl}, \quad \widetilde{m} = \eta^3/\mu^2 = (m_1 m_2 \mu)/(m_1 m_2 - 3\mu^2). \quad (20)$$

It is worthwhile to point out that the Schrödinger-type equation (19), is found to be same as the formula given by Durand and Durand [50]. The perturbation term, $W_{nl}(r)^2$; that is, (v^2/c^2) term in Eqs. (19) and (20) is significant only when it is small (i.e., $W_{nl}(r)/\widetilde{m} \ll 1$). This condition is verified by the confining potentials used to describe the present system except near the color–Coulomb singularity at the origin, and for $r \rightarrow \infty$ (i.e., the wavefunction vanishes at 0 and ∞). However, it is always being satisfied on the average as stated by Ref.[50].

In the present work, we shall study the SS equation with a family of exponential potentials which is called generalized Hulthén potential [51], one of the important molecular potentials, in the $1D$ -vector form,

$$V_q(x) = -V_0 \frac{e^{-\alpha x}}{1 - qe^{-\alpha x}}, \quad V_0 = Ze^2\alpha, \quad 0 \leq x \leq \infty \quad (21)$$

with α denotes the screening (range) parameter, V_0 denotes the coupling constant and q is the deformation parameter which is used to determine the shape of potential. We have to note that, for some specific q values this potential reduces to the well-known types: such as for $q = 0$, to the exponential potential, for $q = 1$ to the standard Hulthén potential, and for $q = -1$ to the Woods-Saxon potential [41]. Further, near the origin, it reduces into the shifted linear potential in the limit of very short range (i.e., $\alpha \rightarrow 0$) [31,32]

$$V_q(x) \approx \frac{V_0}{q-1} + \frac{V_0}{(q-1)^2} \alpha x + O(\alpha^2 x^2), \quad (22)$$

with a constant shift, $V_0/(q-1)$. It also approximates to the screened Coulomb effective potential for small αx (i.e., $\alpha x \rightarrow 0$) as [34]

$$V_{sc}^{eff}(x) \approx -\frac{e^{-\alpha x}}{x} + \frac{\ell(\ell+1)}{2x^2}. \quad (23)$$

The complex form of the potential in (21) is said to be a \mathcal{PT} -Symmetric potential when (cf. Refs.[7,31,32,41])

$$[\mathcal{PT}, V(x)] = 0, \quad (24)$$

i.e., the \mathcal{PT} -symmetry condition for the given potential $V(x)$ satisfies

$$[V(-x)]^* = V(x). \quad (25)$$

To calculate the energy eigenvalues and the corresponding eigenfunctions, the Hermitian real-valued Hulthén potential form given by Eq.(21) is substituted into the 1D \mathcal{PT} -symmetrical Hermitian Schrödinger-type equation (19) for $\ell = 0$ case (i.e., s -wave states):

$$\frac{d^2 \psi_{nq}(x)}{dx^2} + \frac{2\mu}{\hbar^2} \left[E_n + \frac{E_n^2}{2\tilde{m}} + \frac{V_0 e^{-\alpha x}}{(1 - qe^{-\alpha x})} + \frac{V_0^2 e^{-2\alpha x}}{2\tilde{m}(1 - qe^{-\alpha x})^2} + \frac{V_0 E_n e^{-\alpha x}}{\tilde{m}(1 - qe^{-\alpha x})} \right] \psi_{nq}(x) = 0, \quad (26)$$

where $\mu = m/2$ and $\tilde{m} = 2m$ for two identical interacting particles. Now, introducing a convenient dimensionless transformation, $s(x) = e^{-\alpha x}$, satisfying the arbitrary boundary conditions, $0 \leq x \leq \infty \rightarrow 1 \leq s \leq 0$, reduces Eq.(26) to the form:

$$\frac{d^2\psi_{nq}(s)}{ds^2} + \frac{1}{s} \frac{d\psi_{nq}(s)}{ds} + \frac{2\mu}{\hbar^2 \alpha^2 s^2} \left[E_n + \frac{E_n^2}{2\widetilde{m}} + \frac{V_0 s}{(1-qs)} + \frac{V_0^2 s^2}{2\widetilde{m}(1-qs)^2} + \frac{V_0 E_n s}{\widetilde{m}(1-qs)} \right] \psi_{nq}(s) = 0, \quad (27)$$

with the dimensionless definitions given by

$$-\epsilon^2 = \frac{2\mu}{\hbar^2 \alpha^2} \left(E_n + \frac{E_n^2}{2\widetilde{m}} \right) \geq 0 \quad (E_n \leq 0), \quad \epsilon_1 = \frac{2\mu}{\hbar^2 \alpha^2} V_0 \quad (\epsilon_1 > 0),$$

$$\epsilon_2^2 = \frac{2\mu}{\hbar^2 \alpha^2} \frac{V_0^2}{2\widetilde{m}} \quad (\epsilon_2^2 > 0), \quad \epsilon_3 = \frac{2\mu}{\hbar^2 \alpha^2} \frac{V_0 E_n}{\widetilde{m}} \quad (\epsilon_3 > 0) \quad (28)$$

and finally one can arrive at the simple hypergeometric equation given by

$$\psi_{nq}''(s) + \frac{1-qs}{s(1-qs)} \psi_{nq}'(s) + \frac{[s^2(\epsilon_2^2 - q^2\epsilon^2 - q\epsilon_3 - q\epsilon_1) + s(\epsilon_1 + \epsilon_3 + 2q\epsilon^2) - \epsilon^2]}{[s(1-qs)]^2} \psi_{nq}(s) = 0. \quad (29)$$

Hence, comparing the last equation with the generalized hypergeometric type, Eq.(1), we obtain the associated polynomials as

$$\tilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s(1 - qs), \quad \tilde{\sigma}(s) = s^2(\epsilon_2^2 - q^2\epsilon^2 - q\epsilon_3 - q\epsilon_1) + s(\epsilon_1 + \epsilon_3 + 2q\epsilon^2) - \epsilon^2. \quad (30)$$

When these polynomials are substituted into Eq.(7), with $\sigma'(s) = 1 - 2qs$, we obtain

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2} \sqrt{s^2(q^2 + 4(q^2\epsilon^2 + q\epsilon_1 + q\epsilon_3 - \epsilon_2^2) - 4qk) + 4s(k - \epsilon_1 - \epsilon_3 - 2q\epsilon^2) + 4\epsilon^2}. \quad (31)$$

Further, the discriminant of the upper expression under the square root has to be set equal to zero. Therefore, it becomes

$$\Delta = [k - \epsilon_1 - \epsilon_3 - 2q\epsilon^2]^2 - \epsilon^2 [q^2 + 4(q^2\epsilon^2 + q\epsilon_1 + q\epsilon_3 - \epsilon_2^2) - 4qk] = 0. \quad (32)$$

Solving Eq.(32) for the constant k , we obtain the double roots as $k_{+,-} = \epsilon_1 + \epsilon_3 \pm b\epsilon$, where $b = \sqrt{q^2 - 4\epsilon_2^2}$. Thus, substituting these values for each k into Eq.(31), we obtain

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2} \begin{cases} (2q\epsilon - b)s - 2\epsilon; & \text{for } k_+ = \epsilon_1 + \epsilon_3 + b\epsilon, \\ (2q\epsilon + b)s - 2\epsilon; & \text{for } k_- = \epsilon_1 + \epsilon_3 - b\epsilon. \end{cases} \quad (33)$$

Hence, making the following choice for the polynomial $\pi(s)$ as

$$\pi(s) = -\frac{qs}{2} - \frac{1}{2} [(2q\epsilon + b)s - 2\epsilon], \quad (34)$$

for $k_- = \epsilon_1 + \epsilon_3 - b\epsilon$, giving the function:

$$\tau(s) = -q(2 + 2\epsilon + b/q)s + (1 + 2\epsilon), \quad (35)$$

which has a negative derivative of the form $\tau'(s) = -q(2 + 2\epsilon + b/q)$. Thus, from Eqs.(8)-(9) and Eqs.(34)-(35), we find

$$\lambda = -\frac{q}{2}(1 + 2\epsilon)(1 + b/q) + (\epsilon_1 + \epsilon_3), \quad (36)$$

and

$$\lambda_n = (1 + n + 2\epsilon + b/q) nq. \quad (37)$$

Therefore, after setting $\lambda_n = \lambda$ and solving for ϵ , in $\hbar = 1$ units, we find the Salpeter exact binding energy spectra as

$$E_{nq} = \left(\frac{V_0}{2q} - \widetilde{m} \right) \left\{ 1 \pm \sqrt{1 - \frac{2\widetilde{m}a \left[\left(\frac{V_0}{a} \right)^2 - \left(\frac{V_0}{2a} \right) D + \frac{1}{16} D^2 \right]}{\left(\frac{V_0}{2q} - \widetilde{m} \right)^2 D}} \right\}, \quad a = \frac{\hbar^2 \alpha^2}{2\mu}, \quad (38)$$

where

$$D = (C^2 + 4\epsilon_2^2) / q = q + q(2n + 1)^2 + 2(2n + 1)b, \quad C = b + q(2n + 1) \quad (39)$$

where $b = \sqrt{q^2 - \frac{V_0^2}{\alpha^2}}$ for equal mass case. For convenience, when $m_1 = m_2$, the upper expression (38) can be further rearranged as

$$E_{nq} = \left(\frac{V_0}{2q} - 2m \right) \left\{ 1 \pm \sqrt{1 - \frac{(2mV_0)^2 \left[1 - \left(\frac{\xi}{2mqV_0} \right) + \frac{1}{4} \left(\frac{\xi}{2mqV_0} \right)^2 \right]}{\xi \left(\frac{V_0}{2q} - 2m \right)^2}} \right\}, \quad 0 \leq n < \infty, \quad (40)$$

where

$$\xi = q\alpha \left[q\alpha + q\alpha(2n+1)^2 + 2(2n+1)\sqrt{q^2\alpha^2 - V_0^2} \right] = \kappa^2 + V_0^2,$$

$$\kappa = \sqrt{\alpha^2 q^2 - V_0^2} + q\alpha(2n+1), \quad q^2 \geq \left(\frac{V_0}{\alpha} \right)^2. \quad (41)$$

Let us now find the corresponding wavefunctions. Applying the NU method, the hypergeometric function $y_n(s)$ is the polynomial solution of hypergeometric-type equation (3) and described with the weight function [42]. By substituting $\pi(s)$ and $\sigma(s)$ in Eq.(6) and then solving the first-order differential equation, we find

$$\phi(s) = s^\epsilon (1 - qs)^{\frac{(b+q)}{2q}}. \quad (42)$$

It is easy to find the other part of the wave function from the definition of the weight function

$$\rho(s) = s^{2\epsilon} (1 - qs)^{b/q}, \quad (43)$$

which then substituted into the Rodrigues relation resulting in

$$y_{nq}(s) = D_{nq} s^{-2\epsilon} (1 - qs)^{-b/q} \frac{d^n}{ds^n} \left[s^{n+2\epsilon} (1 - qs)^{n+b/q} \right], \quad (44)$$

where D_{nq} is a normalizing constant. In the limit $q \rightarrow 1$, the polynomial solutions of $y_n(s)$ are expressed in terms of Jacobi Polynomials, which is one of the classical orthogonal polynomials, with weight function (43) in the closed interval $[0, 1]$, giving $y_{n,1}(s) \simeq P_n^{(2\epsilon, b)}(1 - 2s)$ [52]. The radial wave function $\psi_{nq}(s)$ is obtained from the Jacobi polynomials in Eq.(44) and $\phi(s)$ in Eq.(42) for the s-wave functions could be determined as

$$\psi_{nq}(s) = N_{nq} s^{-\epsilon} (1 - qs)^{\frac{q-b}{2q}} \frac{d^n}{ds^n} \left[s^{n+2\epsilon} (1 - qs)^{n+b/q} \right] = N_{nq} s^\epsilon (1 - qs)^{\frac{(b+q)}{2q}} P_n^{(2\epsilon, b/q)}(1 - 2qs), \quad (45)$$

with $s = e^{-\alpha x}$ and N_{nq} is a new normalization constant determine by

$$1 = \int_1^0 |\psi_{nq}(s)|^2 ds = N_{nq}^2 \int_1^0 s^{2\epsilon} (1 - qs)^{\frac{(q+b)}{q}} \left[P_n^{(2\epsilon, b/q)}(1 - 2qs) \right]^2 ds, \quad (46)$$

We now make use of the fact that the Jacobi polynomials can be explicitly written in two different ways [52]:

$$P_n^{(\rho, \nu)}(z) = 2^{-n} \sum_{p=0}^n (-1)^{n-p} \binom{n+\rho}{p} \binom{n+\nu}{n-p} (1-z)^{n-p} (1+z)^p, \quad (47)$$

$$P_n^{(\rho, \nu)}(z) = \frac{\Gamma(n+\rho+1)}{n! \Gamma(n+\rho+\nu+1)} \sum_{r=0}^n \binom{n}{r} \frac{\Gamma(n+\rho+\nu+r+1)}{\Gamma(r+\rho+1)} \left(\frac{z-1}{2}\right)^r, \quad (48)$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$. Using Eqs.(47)-(48), we obtain the explicit expressions for $P_n^{(2\epsilon, b/q)}(1-2qs)$

$$P_n^{(2\epsilon, b/q)}(1-2qs) = (-1)^n \Gamma(n+2\epsilon+1) \Gamma(n + \frac{b}{q} + 1) \times \sum_{p=0}^n \frac{(-1)^p q^{n-p}}{p!(n-p)! \Gamma(p + \frac{b}{q} + 1) \Gamma(n+2\epsilon-p+1)} s^{n-p} (1-qs)^p, \quad (49)$$

$$P_n^{(2\epsilon, b/q)}(1-2qs) = \frac{\Gamma(n+2\epsilon+1)}{\Gamma(n+2\epsilon+\frac{b}{q}+1)} \sum_{r=0}^n \frac{(-1)^r q^r \Gamma(n+2\epsilon+\frac{b}{q}+r+1)}{r!(n-r)! \Gamma(2\epsilon+r+1)} s^r. \quad (50)$$

Therefore, substituting Eqs.(49) and (50) into Eq.(46), it gives

$$1 = N_{nq}^2 (-1)^{n+1} \frac{\Gamma(n+\frac{b}{q}+1) \Gamma(n+2\epsilon+1)^2}{\Gamma(n+2\epsilon+\frac{b}{q}+1)} \left\{ \sum_{p=0}^n \frac{(-1)^p q^{n-p}}{p!(n-p)! \Gamma(p+\frac{b}{q}+1) \Gamma(n+2\epsilon-p+1)} \right\} \times \left\{ \sum_{r=0}^n \frac{(-1)^r q^r \Gamma(n+2\epsilon+\frac{b}{q}+r+1)}{r!(n-r)! \Gamma(2\epsilon+r+1)} \right\} I_{nq}(p, r), \quad (51)$$

where

$$I_{nq}(p, r) = \int_0^1 s^{n+2\epsilon+r-p} (1-qs)^{p+\frac{b}{q}+1} ds. \quad (52)$$

Using the following integral representation of the hypergeometric function [53]

$$\int_0^1 s^{\alpha_0-1} (1-s)^{\gamma_0-\alpha_0-1} (1-qs)^{-\beta_0} ds = {}_2F_1(\alpha_0, \beta_0 : \gamma_0; q) \frac{\Gamma(\alpha_0) \Gamma(\gamma_0 - \alpha_0)}{\Gamma(\gamma_0)},$$

$$[\operatorname{Re}(\gamma_0) > \operatorname{Re}(\alpha_0) > 0, \quad |\arg(1-q)| < \pi] \quad (53)$$

which gives

$${}_2F_1(\alpha_0, \beta_0 : \alpha_0 + 1; q)/\alpha_0 = \int_0^1 s^{\alpha_0-1} (1 - qs)^{-\beta_0} ds. \quad (54)$$

The hypergeometric function ${}_2F_1(\alpha_0, \beta_0 : \gamma_0; 1)$ reduces into

$${}_2F_1(\alpha_0, \beta_0 : \gamma_0; 1) = \frac{\Gamma(\gamma_0)\Gamma(\gamma_0 - \alpha_0 - \beta_0)}{\Gamma(\gamma_0 - \alpha_0)\Gamma(\gamma_0 - \beta_0)},$$

$$[\text{Re}(\gamma_0 - \alpha_0 - \beta_0) > 0, \text{Re}(\gamma_0) > \text{Re}(\beta_0) > 0], \quad (55)$$

for $q = 1$. Setting $\alpha_0 = n + 2\epsilon + r - p + 1$, $\beta_0 = -p - \frac{b}{q} - 1$, and $\gamma_0 = \alpha_0 + 1$, one gets

$$I_{nq}(p, r) = \frac{{}_2F_1(\alpha_0, \beta_0 : \gamma_0; q)}{\alpha_0} = \frac{(n + 2\epsilon + r - p + 1)!(p + \frac{b}{q} + 1)!}{(n + 2\epsilon + r - p + 1)(n + 2\epsilon + r + \frac{b}{q} + 2)!}. \quad (56)$$

A. Real potentials

Firstly, we consider the real case in Eq.(21), i.e., all parameters (V_0, q, α) are real:

(i) For any given α the spectrum consists of real eigenstate spectra $E_n(V_0, q, \alpha)$ depending on q . The sign of V_0 does not affect the bound states. It is clear that while $V_0 \rightarrow 0$, $E_n = -2m \left[1 \pm \sqrt{1 - \left(\frac{(n+1)\alpha}{2m} \right)^2} \right]$ which is for the ground state (i.e., $n = 0$) tend to the value $E_0 \approx -3.73m$ and for the first excited state (i.e., $n = 1$) takes the value $E_1 = -2m$, where we have used $\lambda_c = \frac{\hbar}{mc} = \frac{1}{m} = \frac{1}{\alpha}$ which is the compton wavelength of the Salpeter particles.

(ii) There exist bound states (real solution) in case if the condition $\frac{V_0^2}{\alpha^2} \leq q^2$ is achieved, otherwise there are no bound-states.

(iii) There exist bound states in case if the condition $(2mV_0)^2 \left[1 - \frac{\xi}{2mqV_0} + \frac{1}{4} \left(\frac{\xi}{2mqV_0} \right)^2 \right] \leq \left(\frac{V_0}{2q} - 2m \right)^2 \xi$, with $\xi = \kappa^2 + V_0^2$ is achieved, otherwise there are no bound-states.

Moreover, this condition which gives the critical coupling value turns to be

$$n \leq \frac{1}{2q\alpha} \left(\sqrt{\chi^2 - V_0^2} - \sqrt{q^2\alpha^2 - V_0^2} \right) - \frac{1}{2}, \quad (57)$$

where

$$\chi^2 = 2q^2 \left\{ \left[\frac{2mV_0}{q} + \left(\frac{V_0}{2q} - 2m \right)^2 \right] \pm \left(\frac{V_0}{2q} - 2m \right) \sqrt{\left(\frac{V_0}{2q} - 2m \right)^2 + \frac{4mV_0}{q}} \right\} \quad (58)$$

i.e., there are only finitely many eigenstates. In order that at least one level might exist, its necessary that the inequality

$$q\alpha + \sqrt{q^2\alpha^2 - V_0^2} \leq \sqrt{\chi^2 - V_0^2}, \quad (59)$$

is fulfilled

For a more specific case $q = -1$, Eq.(21) is reduced into the shifted Woods-Saxon (WS) potential

$$V(x) = -V_0 + \frac{V_0}{1 + e^{-\alpha x}}, \quad (60)$$

with energy spectrum given by

$$E_n(V_0, \alpha) = - \left(\frac{V_0}{2q} + 2m \right) \left\{ 1 \pm \sqrt{1 - \frac{(2mV_0)^2}{\tilde{\xi}} \frac{\left[1 + \left(\frac{\tilde{\xi}}{2mV_0} \right) + \frac{1}{4} \left(\frac{\tilde{\xi}}{2mV_0} \right)^2 \right]}{\left(\frac{V_0}{2} + 2m \right)^2}} \right\}, \quad 0 \leq n < \infty, \quad (61)$$

where

$$\tilde{\xi} = \alpha \left[\alpha + \alpha(2n+1)^2 - 2(2n+1)\sqrt{\alpha^2 - V_0^2} \right], \quad \alpha^2 \geq V_0^2. \quad (62)$$

In this case, for any given α , all the eigenstates $E_n \leq 0$.

(iv) For $q = 0$, the potential (21) reduces into the exponential form

$$V(x) = -V_0 e^{-\alpha x}, \quad (63)$$

the energy expression (40) does not give an explicit form, i.e., the NU method fails to give energy expression to this type of exponential potential. It is noted that for this potential there is no explicit form of energy expression of bound states for Schrödinger [16], KG [31,38] and also Dirac [36] equations.

For $q = 0$, the generalized equation of hypergeometric type which is given by Eq.(29) becomes

$$\psi_n''(s) + \frac{1}{s}\psi_n'(s) + \frac{[s^2\epsilon_2^2 + s(\epsilon_1 + \epsilon_3) - \epsilon^2]}{s^2}\psi_n(s) = 0, \quad (64)$$

with

$$\tilde{\tau}(s) = 1, \quad \sigma(s) = s, \quad \tilde{\sigma}(s) = s^2\epsilon_2^2 + s(\epsilon_1 + \epsilon_3) - \epsilon^2, \quad (65)$$

and the corresponding $\pi(s)$ is determined as

$$\pi(s) = \pm \begin{cases} i\epsilon_2 s + \epsilon; & \text{for } k_+ = \epsilon_1 + \epsilon_3 + 2i\epsilon_2\epsilon, \\ i\epsilon_2 s - \epsilon; & \text{for } k_- = \epsilon_1 + \epsilon_3 - 2i\epsilon_2\epsilon, \end{cases} \quad (66)$$

where $i = \sqrt{-1}$ and $\epsilon_2 = \frac{V_0}{2\alpha}$. Following a procedure similar to the previous case, when $\pi(s) = -i\epsilon_2 s + \epsilon$ is chosen for $k = \epsilon_1 + \epsilon_3 - 2i\epsilon_2\epsilon$, then

$$\tau(s) = -2i\epsilon_2 s + (1 + 2\epsilon), \quad (67)$$

and

$$\lambda = \epsilon_1 + \epsilon_3 - i\epsilon_2 - 2i\epsilon_2\epsilon, \quad \lambda_n = 2ni\epsilon_2, \quad \phi_n(s) = s^\epsilon e^{-i\epsilon_2 s}, \quad (68)$$

could be obtained. Substituting $\sigma(s)$ and $\tau(s)$, together with λ into Eq.(3) gives

$$sy_n''(s) + [1 + 2\epsilon - 2i\epsilon_2 s]y_n'(s) - [i\epsilon_2 + 2i\epsilon_2\epsilon - (\epsilon_1 + \epsilon_3)]y_n(s) = 0. \quad (69)$$

The last equation can also be reduced to standard Whittaker differential equation [54]. Thus, the solutions vanishing at infinity and it can be written in terms of confluent hypergeometric function as follows:

$$y_n(s) = {}_1F_1\left(\frac{1}{2} + \epsilon + i\frac{(\epsilon_1 + \epsilon_3)}{2\epsilon_2}; 1 + 2\epsilon; 2i\epsilon_2 s\right). \quad (70)$$

So, the acceptable solution for the upper component is found to be

$$\psi_n(s) = \phi(s)y_n(s) = A_1 F_1\left(\frac{1}{2} + \epsilon + i\frac{(\epsilon_1 + \epsilon_3)}{2\epsilon_2}; 1 + 2\epsilon; 2i\epsilon_2 s\right) s^\epsilon e_1^{-i\epsilon_2 s}. \quad (71)$$

On the other hand, to find the an expression for the exact energy spectrum, when $\lambda_n = \lambda$, in $\hbar = 1$ units, we obtain

$$E_n(\alpha) = i\frac{2n+1}{2}\alpha - 2m - i\frac{2m^2}{(2n+1)\alpha}, \quad 0 \leq n < \infty. \quad (72)$$

From the last equation we conclude that if and only if $i\alpha$ is real then E_n is a real. Hence, in order to apply NU method to this type of exponential potential ϵ_2 should be complex (imaginary). This leads to result that either V_0 or α must be imaginary. Therefore, for purely imaginary α , i.e., $\alpha \rightarrow i\alpha_I$, it reads:

$$E_n(\alpha) = -m \left[2 + \frac{(2n+1)}{2m}\alpha_I + \frac{2m}{(2n+1)\alpha_I} \right], \quad 0 \leq n < \infty. \quad (73)$$

In the nonrelativistic limit, there is no available bound-state energy solution for the exponential potentials [16].

B. Complex potentials

Let us consider the case where at least one of the potential parameters be complex:

(I) If α is a complex parameter ($\alpha \rightarrow i\alpha$), the potential (21) becomes

$$V_q(x) = \frac{V_0}{q^2 - 2q \cos(\alpha x) + 1} [q - \cos(\alpha x) + i \sin(\alpha x)] = V_q^*(-x), \quad (74)$$

which is a \mathcal{PT} -symmetric but non-Hermitian. It has real spectrum given by

$$E_n(V_0, i\alpha, q) = \left(\frac{V_0}{2q} - 2m \right) \left\{ 1 \pm \sqrt{1 + \frac{(2mV_0)^2 \left[1 + \left(\frac{\varsigma}{2mqV_0} \right) + \frac{1}{4} \left(\frac{\varsigma}{2mqV_0} \right)^2 \right]}{\varsigma \left(\frac{V_0}{2q} - 2m \right)^2}} \right\},$$

with

$$\varsigma = q^2\alpha^2 + q^2\alpha^2(2n+1)^2 + 2q\alpha(2n+1)\sqrt{q^2\alpha^2 + V_0^2}, \quad (75)$$

if and only if $-(2mV_0)^2 \left[1 + \left(\frac{\varsigma}{2mqV_0} \right) + \frac{1}{4} \left(\frac{\varsigma}{2mqV_0} \right)^2 \right] \leq \varsigma \left(\frac{V_0}{2q} - 2m \right)^2$. The corresponding radial wave function $\psi_{nq}(s)$ for the s-wave could be determined as

$$\psi_{nq}(s) = N_{nq} s^{i\epsilon} (1 - qs)^{\frac{(c+q)}{2q}} P_n^{(2i\epsilon, c/q)}(1 - 2qs), \quad (76)$$

where $c = \sqrt{q^2 + \frac{V_0^2}{\alpha^2}}$, and $s = e^{-i\alpha x}$.

The norm of the wavefunction of such a non-Hermitian quantum mechanical system is redefined as [2,55]

$$\int_0^\infty \psi_{nq}^*(-s) \psi_{nq}(s) ds = \nu, \quad \nu = \pm 1. \quad (77)$$

$\nu = 1$ stands for \mathcal{PT} -symmetric phase whereas $\nu = -1$ stands for \mathcal{PT} -antisymmetric phase. Therefore making the necessary parameter replacements in Eqs.(51)-(52), we can obtain the normalization constant for the complex \mathcal{PT} -symmetric generalized Hulthén potential given by Eq.(74).

For the sake of comparing the relativistic and non-relativistic binding energies, we need to solve the 1D Schrödinger equation for the complex generalized Hulthén potential. We set the convenient transformation $s(x) = e^{-i\alpha x}$, $0 \leq x \leq \infty \rightarrow 1 \leq s \leq 0$, to obtain

$$\psi_{nq}''(s) + \frac{1 - qs}{(s - qs^2)} \psi_{nq}'(s) + \frac{[s^2(q\tilde{\epsilon}_1 - q^2\tilde{\epsilon}^2) + s(2q\tilde{\epsilon}^2 - \tilde{\epsilon}_1) - \tilde{\epsilon}^2]}{(s - qs^2)^2} \psi_{nq}(s) = 0, \quad (78)$$

for which

$$\tilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s - qs^2, \quad \tilde{\sigma}(s) = s^2(q\tilde{\epsilon}_1 - q^2\tilde{\epsilon}^2) + s(2q\tilde{\epsilon}^2 - \tilde{\epsilon}_1) - \tilde{\epsilon}^2,$$

$$\tilde{\epsilon}^2 = \frac{2\mu}{\hbar^2 \alpha^2} E_n, \quad \tilde{\epsilon}_1 = \frac{2\mu}{\hbar^2 \alpha^2} V_0 \quad (\tilde{\epsilon}_1 > 0). \quad (79)$$

Moreover, it could be obtained

$$\tau(s) = -q(3 + 2\tilde{\epsilon})s + (1 + 2\tilde{\epsilon}), \quad (80)$$

if $\pi(s) = -q(1 + \tilde{\epsilon})s + \tilde{\epsilon}$ is chosen for $k_- = -q\tilde{\epsilon} - \tilde{\epsilon}_1$. Finally, the bound-state spectra in the non-relativistic limit could be found as

$$E_n(V_0, q, i\alpha) = \frac{\hbar^2}{8\mu q^2 \alpha^2} \left[\frac{2\mu V_0 / \hbar^2 + q\alpha^2(n+1)^2}{(n+1)} \right]^2 > 0, \quad 0 \leq n < \infty. \quad (81)$$

On the other hand, for the sake of comparison, the non-relativistic limit for the real potential (21) can be found directly from the last equation as

$$E_n(V_0, q, \alpha) = -\frac{\hbar^2 \alpha^2}{8\mu} \left[(n+1) - \frac{\beta}{(n+1)} \right]^2, \quad 0 \leq n < \infty. \quad (82)$$

where $\beta = \frac{2\mu V_0}{q\alpha^2 \hbar^2}$. The radial wave function in the current case becomes

$$\psi_{nq}(s) = N_{nq} s^{\tilde{\epsilon}} (1 - qs) P_n^{(2\tilde{\epsilon}, 1)}(1 - 2qs), \quad (83)$$

with $s = e^{-i\alpha x}$ and N_{nq} is a new normalization constant determined by

$$1 = N_{nq}^2 (-1)^{n+1} \frac{(n+1)! \Gamma(n+2\tilde{\epsilon}+1)^2}{\Gamma(n+2\tilde{\epsilon}+2)} \left\{ \sum_{p=0}^n \frac{(-1)^p q^{n-p}}{p!(n-p)!(p+1)! \Gamma(n+2\tilde{\epsilon}-p+1)} \right\} \\ \times \left\{ \sum_{r=0}^n \frac{(-1)^r q^r \Gamma(n+2\tilde{\epsilon}+r+2)}{r!(n-r)! \Gamma(2\tilde{\epsilon}+r+1)} \right\} \int_0^1 s^{n+2\tilde{\epsilon}+r-p} (1 - qs)^{p+2} ds, \quad (84)$$

where

$$\int_0^1 s^{n+2\tilde{\epsilon}+r-p} (1 - qs)^{p+2} ds = {}_2F_1(n+2\tilde{\epsilon}+r-p+1, -p-2; n+2\tilde{\epsilon}+r-p+2; q) B(n+2\tilde{\epsilon}+r-p+1, 1), \quad (85)$$

(II) Let two parameters; namely, V_0 and q be complex parameters (i.e., $V_0 \rightarrow iV_0$, $q \rightarrow iq$), then the potential transforms to the form

$$V_q(x) = V_0 \frac{[2 \cosh^2(\alpha x) - \sinh(2\alpha x) - 1] - i [\cosh(\alpha x) - \sinh(\alpha x)]}{1 + q^2 [2 \cosh^2(\alpha x) - \sinh(2\alpha x) - 1]}, \quad (86)$$

which is a \mathcal{PT} -symmetric but non-Hermitian if $(2mV_0)^2 \left[1 - \left(\frac{\xi}{2mqV_0} \right) + \frac{1}{4} \left(\frac{\xi}{2mqV_0} \right)^2 \right] \leq \xi \left(\frac{V_0}{2q} - 2m \right)^2$, it may possess real spectra as

$$E_n(iV_0, \alpha, iq) = \left(\frac{V_0}{2q} - 2m \right) \left\{ 1 \pm \sqrt{1 - \frac{(2mV_0)^2 \left[1 - \left(\frac{\xi}{2mqV_0} \right) + \frac{1}{4} \left(\frac{\xi}{2mqV_0} \right)^2 \right]}{\xi \left(\frac{V_0}{2q} - 2m \right)^2}} \right\},$$

where ξ is defined by Eq.(41). On the other hand, the corresponding radial wave functions $\psi_{nq}(s)$ for the s-wave could be determined as

$$\psi_{nq}(s) = N_{nq}s^\epsilon(1 - iqs)^{\frac{(d+q)}{2q}}P_n^{(2\epsilon, d/q)}(1 - 2iqs), \quad (87)$$

with $s = e^{-\alpha x}$ and $d = \sqrt{q^2 - \frac{V_0^2}{\alpha^2}}$. The integral $I_{nq}(p, r) = \int_0^1 s^{n+2\epsilon+r-p}(1 - iqs)^{p+\frac{d}{q}+1} ds$ is given by

$$I_{nq}(p, r) = {}_2F_1(n + 2\epsilon + r - p + 1, -p - \frac{d}{q} - 1 : n + 2\epsilon + r - p + 2; iq)B(n + 2\epsilon + r - p + 1, 1). \quad (88)$$

(III) When all the parameters V_0 , α and q are complex parameters (i.e., $V_0 \rightarrow iV_0$, $\alpha \rightarrow i\alpha$, $q \rightarrow iq$), we obtain

$$V_q(x) = \frac{V_0}{q^2 - 2q \sin(\alpha x) + 1} [q - \sin(\alpha x) - i \cos(\alpha x)] = V_q^*(\frac{\pi}{2} - x). \quad (89)$$

This potential is a pseudo-Hermitian potential [27,56] having a $\pi/2$ phase difference with respect to the potential in case (I), it is also a \mathcal{PT} -symmetric, $\eta = P$ -pseudo-Hermitian (i.e., $PTV_q(x)(PT)^{-1} = V_q(x)$, with $P = \eta : x \rightarrow \frac{\pi}{2\alpha} - x$ and $T : i \rightarrow -i$) but non-Hermitian having real spectrum given by

$$E_n(iV_0, i\alpha, iq) = \left(\frac{V_0}{2q} - 2m \right) \left\{ 1 \pm \sqrt{1 + \frac{(2mV_0)^2}{\tilde{\zeta}} \left[1 + \left(\frac{\tilde{\zeta}}{2mqV_0} \right) + \frac{1}{4} \left(\frac{\tilde{\zeta}}{2mqV_0} \right)^2 \right]} \right\}, \quad (90)$$

where

$$\tilde{\zeta} = q^2\alpha^2 + q^2\alpha^2(2n+1)^2 - 2q\alpha(2n+1)\sqrt{q^2\alpha^2 + V_0^2}. \quad (91)$$

On the other hand, the corresponding radial wave functions $\psi_{nq}(s)$ for the s-wave could be determined as

$$\psi_{nq}(s) = N_{nq}s^{i\epsilon}(1 - iqs)^{\frac{(c+q)}{2q}}P_n^{(2i\epsilon, \tilde{c}/q)}(1 - 2iqs), \quad (92)$$

with $s = e^{-i\alpha x}$ and c is defined after Eq.(76). The integral $I_{nq}(p, r) = \int_0^1 s^{n+2i\epsilon+r-p}(1 - iqs)^{p+\frac{c}{q}+1} ds$ is given by

$$I_{nq}(p, r) = {}_2F_1(n + 2i\epsilon + r - p + 1, -p - \frac{c}{q} - 1 : n + 2i\epsilon + r - p + 2; iq)B(n + 2i\epsilon + r - p + 1, 1). \quad (93)$$

IV. RESULTS AND CONCLUSIONS

We have seen that the s -wave Salpeter equation for the generalized Hulthén potential can be solved exactly. The relativistic bound-state energy spectrum and the corresponding wave functions for the Hulthén potential have been obtained by the NU method. Some interesting results including the \mathcal{PT} -symmetric and pseudo-Hermitian versions of the generalized Hulthén potential have also been discussed for the real bound-states. In addition, we have discussed the relation between the non-relativistic and relativistic solutions and the possibility of existence of bound states for complex parameters. Finally, we have shown the possibility to obtain relativistic bound-states of complex quantum mechanical formulations.

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